NAMIBIA UNIVERSITY
OF SCIENCE AND TECHNOLOGY

FACULTY OF HEALTH, APPLIED SCIENCES AND NATURAL RESOURCES

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUALIFICATION: Bachelor of science in Applied Mathematics and Statistics	
QUALIFICATION CODE: 07BAMS	**LEVEL:** 6
COURSE CODE: PBT602S	**COURSE NAME:** Probability Theory 2
SESSION: JUNE 2022	**PAPER:** THEORY
DURATION: 3 HOURS	**MARKS:** 100

FIRST OPPORTUNITY EXAMINATION QUESTION PAPER	
EXAMINER	Dr D. B. GEMECHU
MODERATOR:	Prof R. KUMAR

INSTRUCTIONS

1. There are 5 questions, answer ALL the questions by showing all the necessary steps.
2. Write clearly and neatly.
3. Number the answers clearly.
4. Round your answers to at least four decimal places, if applicable.

PERMISSIBLE MATERIALS

1. Nonprogrammable scientific calculator

THIS QUESTION PAPER CONSISTS OF 4 PAGES (Including this front page)

Question 1 [12 marks]

1.1. Define the following terms:

1.1.1. Probability function [3]

1.1.2. Power set [1]

1.1.3. σ–algebra [2]

1.1.4. Consider an experiment of rolling a die with six faces once.

1.1.4.1. Show that the set $\sigma(X) = \{\phi, S, \{1,2,4\}, \{3,5,6\}\}$ is a sigma algebra, where S represents the sample space for a random experiment of rolling a die with six faces. [3]

1.1.4.2. Given a set $Y = \{\{1,2,3,5\}, \{4\}, \{6\}\}$, then generate the smallest sigma algebra, $\sigma(Y)$ that contains a set Y. [3]

Question 2 [24 marks]

2.1. Let X be a continuous random variable with p.d.f. given by

$$f(x) = \begin{cases} x+1, & for\ -1 < x < 0, \\ 1-x, & for\ 0 \le x < 1, \\ 0, & otherwise. \end{cases}$$

Then find cumulative density function of X [7]

2.2. Suppose that the joint CDF of a two dimensional continuous random variable is given by

$$F_{XY}(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, & if\ x > 0;\ y > 0, \\ 0, & otherwise. \end{cases}$$

Then find the joint p.d.f. of X and Y. [4]

2.3. Consider the following joint pdf of X and Y. [7]

$$f(x,y) = \begin{cases} 2, & x > 0;\ y > 0;\ x + y < 1, \\ 0, & elsewhere. \end{cases}$$

2.3.1. Find the marginal probability density function of $f(y)$ [2]

2.3.2. Find the conditional probability density function of X given $Y = y$, $f_X(x|Y = y)$ [2]

2.3.3. Find $P\left(X < \frac{1}{2} \middle| Y = \frac{1}{4}\right)$ [3]

2.4. Let Y_1, Y_2, and Y_3 be three random variables with $E(Y_1) = 2$, $E(Y_2) = 3$, $E(Y_3) = 2$, $\sigma_{Y_1}^2 = 2$, $\sigma_{Y_2}^2 = 3$, $\sigma_{Y_3}^2 = 1$, $\sigma_{Y_1Y_2} = -0.6$, $\sigma_{Y_1Y_3} = 0.3$, and $\sigma_{Y_2Y_3} = 2$.

2.4.1. Find the expected value and variance of $U = 2Y_1 - 3Y_2 + Y_3$ [2]

2.4.2. If $W = Y_1 + 2Y_3$, find the covariance between U and W [4]

QUESTION 3 [27 marks]

3.1. Let X be a discrete random variable with a probability mass function $P(x)$, then show that the moment generating function of X is a function of all the moments μ_k' about the origin which is given by

$$M_X(t) = E(e^{tx}) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots . + \frac{t^k}{k!}\mu_k' + \cdots$$

Hint: use Taylor's series expansion: $e^{tx} = \sum_{i=1}^{\infty} \frac{(tx)^i}{i!}$ [5]

3.2. Let $X_1, X_2, \dots, X_n$ be a random sample from a Gamma distribution with parameters α and θ, that is

$$f(x_i|\alpha,\theta) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^\alpha} x_i^{\alpha-1} e^{-\frac{x_i}{\theta}}, & for\ x_i > 0;\ \alpha > 0;\ \theta > 0, \\ 0 & otherwise. \end{cases}$$

3.2.1. Show that the moment generating function of X_i is given by $M_{X_i}(t) = \left(\frac{1}{1-\theta t}\right)^\alpha$ [6]

3.2.2. Find the mean of X using the moment generating function of X. [4]

3.3. Suppose that X is a random variable having a binomial distribution with the parameters n and p (i.e., $X \sim Bin(n,p)$)

3.3.1. Find the cumulant generating function of X and find the first cumulant.
Hint: $M_X(t) = \left(1 - p(1-e^t)\right)^n$ [4]

3.3.2. If we define another random variable $Y = aX + b$, then derive the moment generating function of Y, where a and b be any constant numbers. [3]

3.4. Let X and Y be two continuous random variables with $f(x)$ and $g(y)$ be a pdf of X and Y, respectively, then show that $E[g(y)] = E\big[E[g(y)|X]\big]$. [5]

Question 4 [20 marks]

4.1. Suppose that X and Y are two independent random variables following a chi-square distribution with m and n degrees of freedom, respectively. Use the moment generating function to show that $X + Y \sim \chi^2(m+n)$. (**Hint:** $M_X(t) = \left(\frac{1}{1-2t}\right)^{\frac{m}{2}}$). [7]

4.2. If $X \sim Poisson\ (\lambda)$, find $E(X)$ and $Var(X)$ using the characteristic function of X.

4.2.1. Show that the characteristic function of X is given by $\phi_X(t) = e^{\lambda(e^{it}-1)}$ [5]

4.2.2. Find $E(X)$ and $Var(X)$ using the characteristic function of X. [8]

QUESTION 5 [17 marks]

5.1. Let X_1 and X_2 be independent random variables with the joint probability density function given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & if\ x_1 > 0;\ x_2 > 0, \\ 0, & otherwise. \end{cases}$$

Find the joint probability density function of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$ [10]

5.2. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2. Use the convolution formula to show that $X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. [7]

=== END OF PAPER===

TOTAL MARKS: 100